

From Kepler's Laws to Newtonian Motion and the Direction Angle of Hamilton's Hodograph

Dr. Klaus Huber
Huber Consult
Sesenheimer Str. 21
10627 Berlin
Germany
email: email@klaus-huber.net

Abstract

In this contribution it is shown that the path from Kepler's results to Newtonian motion can be remarkably short and simple. Following this path we also give a straight forward computation of the direction angle of Hamilton's Hodograph. Then we show how the speed as function of the direction angle can be expressed and inverted elegantly using elliptic functions.

Index Terms — Kepler's Laws, Newtonian Motion, Inverse Square Law, Hamilton's Hodograph, Gauss transform.

1 Introduction

Usually Newtonian motion of planets under an inverse square law force is established using Kepler's second law. Assuming at the outset that the inverse square law rules elliptic motion results (or hyperbolic and parabolic motion).

Here we use Kepler's first and second law to find the inverse square law. This procedure also gives the opportunity to find a simple formula for the direction angle of Hamilton's Hodograph.

2 Kepler's laws

We start ab ovo with Kepler's three laws:

- i. A planet moves on an ellipse around the sun with the sun placed in a focus of the ellipse.
- ii. The straight line from the sun to a planet sweeps out equal areas in equal times.
- iii. The time period T for a revolution and the major semiaxis a of the ellipse behave as $T^2 = \text{const} \cdot a^3$.

3 The shortest path from Kepler's laws to Newtonian motion

All motion happens in a plane. Without loss of generality we consider an ellipse with reference to perihel (i.e. the position where the distance of the planet is closest to the sun). The radius is then given by

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} .$$

The above parameters are: the major semiaxis a , the excentricity of the ellipse e , and the angle θ called true anomaly in astronomy. It is convenient to use complex numbers with $\imath = \sqrt{-1}$. The position in the complex plane is given by

$$Z = \frac{a(1 - e^2)}{1 + e \cos \theta} \exp(\imath\theta) .$$

The derivative with respect to time t , denoted by a dot, then immediately gives the velocity

$$\dot{Z} = \frac{a(1 - e^2)\dot{\theta}}{(1 + e \cos \theta)^2} \{\imath e + \imath \exp(\imath\theta)\} .$$

From Kepler's second law we find $\frac{1}{2}r^2\dot{\theta} = \frac{\pi ab}{T}$ where πab is the area of an ellipse with major- and minor semiaxes a and b and T is the time for a complete revolution. As $b = a\sqrt{1 - e^2}$ we get $\dot{\theta} = \frac{2\pi}{T(1 - e^2)^{\frac{3}{2}}} (1 + e \cos \theta)^2$ and

$$\dot{Z} = \frac{2\pi a}{T} \frac{1}{\sqrt{1 - e^2}} \{\imath e + \imath \exp(\imath\theta)\} . \quad (1)$$

We easily see that $\dot{Z}$ describes a circle in the complex plane. Taking the second derivative yields the acceleration

$$\ddot{Z} = -\frac{4\pi^2 a^3}{T^2} \frac{(1 + e \cos \theta)^2}{a^2(1 - e^2)^2} \exp(\imath\theta) = -\frac{4\pi^2 a^3}{T^2} \frac{1}{r^2} \exp(\imath\theta) ,$$

which, setting $\frac{4\pi^2 a^3}{T^2} = G$, leads to the inverse square law force for an object of mass m :

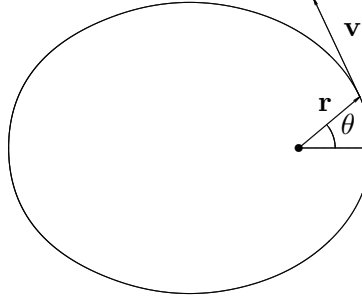
$$m\ddot{Z} = -\frac{mG}{r^2} \exp(\imath\theta) .$$

Note that Kepler's third law $\frac{4\pi^2 a^3}{T^2} = G$ is obtained automatically.

4 Hamilton's Hodograph

Let $\mathbf{r}$ be the location vector for the orbit of a planet (see figure 1). Then $\mathbf{v} = \frac{d}{dt}\mathbf{r}$ gives the velocity vector, i.e. the vector whose length $v = |\mathbf{v}|$ gives the speed. If we consider all vectors $\mathbf{v}$ on the curve which $\mathbf{r}$ reaches and if we translate all vectors to the same point (e.g. to the focus for Newtonian motion) the endpoints of the velocity vectors give rise to a curve which is called the Hodograph.

Figure 1: Elliptic orbit having $e = 0.6$



Hamilton introduced the Hodograph in [2] and found out that for Newtonian motion according to an inverse square law force the hodograph is a circle. A look at the equation for $\dot{Z}$ immediately shows that it describes the hodograph. The only task left is to express $\dot{Z}$ using the angle ϑ_H which goes from the perihel to the vector $\mathbf{v}$, the subscript H stands for Hamilton. This is displayed in fig. 2.

For convenience we use the angle $\vartheta = \vartheta_H - \frac{\pi}{2}$ instead of ϑ_H and find ¹

$$\tan \vartheta = \frac{\sin \theta}{e + \cos \theta} . \quad (2)$$

To find θ as function of ϑ we express $\dot{Z}$ using ϑ which gives

$$\dot{Z} = \frac{2\pi a}{T} \frac{1}{\sqrt{1-e^2}} \left(\sqrt{1-e^2 \sin^2(\vartheta)} + e \cos \vartheta \right) \cdot i \exp(i\vartheta) . \quad (3)$$

Equating with (1) and separating real- and imaginary parts gives

$$\begin{aligned} \sin \theta &= (\sqrt{1-e^2 \sin^2(\vartheta)} + e \cos \vartheta) \sin \vartheta \\ e + \cos \theta &= (\sqrt{1-e^2 \sin^2(\vartheta)} + e \cos \vartheta) \cos \vartheta \end{aligned}$$

which yields

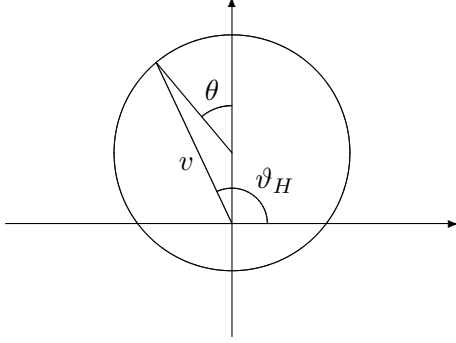
$$\tan \theta = \frac{\sin \vartheta (\sqrt{1-e^2 \sin^2 \vartheta} + e \cos \vartheta)}{\cos \vartheta (\sqrt{1-e^2 \sin^2 \vartheta} + e \cos \vartheta) - e} . \quad (4)$$

Clearly the speed is given by

$$v = \frac{2\pi a}{T} \frac{1}{\sqrt{1-e^2}} \left(\sqrt{1-e^2 \sin^2(\vartheta)} + e \cos \vartheta \right) . \quad (5)$$

¹Here and at subsequent places the reader should have no problems to resolve any ambiguities when applying inverse functions like the arctan-function.

Figure 2: Hodograph $\dot{Z} = v \cdot \exp(i \vartheta_H)$ for $e = 0.6$



5 Computing ϑ from v

The speed v as function of ϑ is given in (5). We now invert this function, i.e. we solve for ϑ . This can be done directly using elliptic functions². We set $\vartheta = \text{am}(u, k)$, where $\text{am}(u, k)$ is Jacobi's amplitude function, which has as input u and the modulus k with $k = e$ and get

$$v = \frac{2\pi a}{T} \frac{1}{\sqrt{1-e^2}} (\text{dn}(u, e) + e \cdot \text{cn}(u, e)) .$$

In appendix A it is shown that this can be expressed using the dn-function alone to give

$$v = \frac{2\pi a}{T} \sqrt{\frac{1+e}{1-e}} \cdot \text{dn} \left(\frac{1+e}{2} u, \frac{2\sqrt{e}}{1+e} \right) . \quad (6)$$

Hence we immediately can express ϑ as function of v .

$$\vartheta = \text{am} \left(\frac{\text{dn}^{-1} \left(\frac{v}{\frac{2\pi a}{T} \sqrt{\frac{1-e}{1+e}}}, \frac{2\sqrt{e}}{1+e} \right)}{(1+e)/2}, e \right) . \quad (7)$$

Note that all involved functions and inverse functions can be easily and efficiently computed (in addition they are included in all relevant computer algebra programs).

6 The angle between $\mathbf{r}$ and $\mathbf{v}$

Having determined ϑ_H (or ϑ) we immediately get the angle φ between $\mathbf{r}$ and $\mathbf{v}$ as

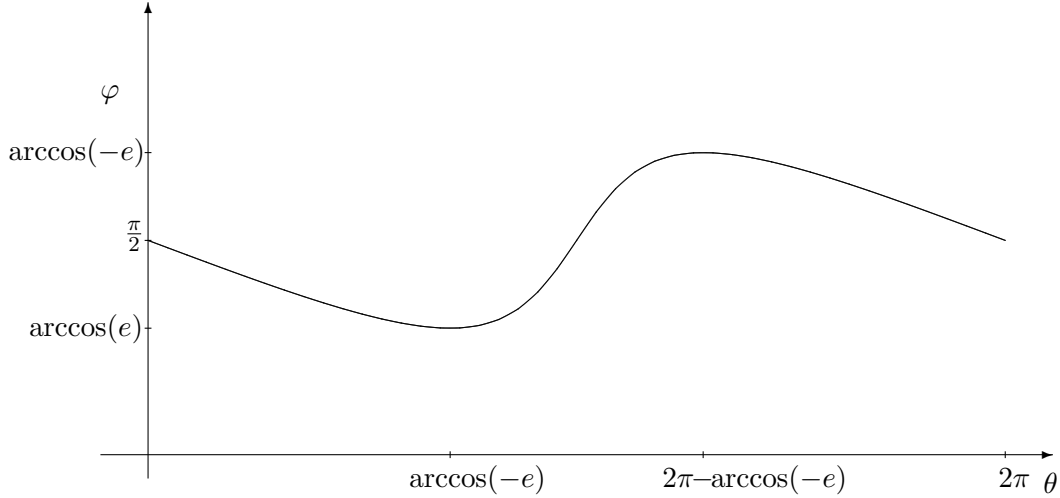
$$\varphi = \vartheta_H - \theta = \frac{\pi}{2} + \vartheta - \theta \Rightarrow \varphi = \frac{\pi}{2} + \arctan \left(\frac{\sin \theta}{e + \cos \theta} \right) - \theta . \quad (8)$$

²Some basic knowledge of Elliptic functions is assumed, see e.g. the book of Whittaker and Watson [3]. For the definition of the functions sn, cn, and dn see appendix A.

The angle φ is displayed as function of θ in figure 3 for $e = 0.6$. Clearly, at perihel (at $\theta = 0$) and aphel (at $\theta = \pi$) this angle equals $\frac{\pi}{2}$. For $0 < \theta < \pi$ the angle φ is smaller than $\pi/2$ and in the interval $\pi < \theta < 2\pi$ the angle φ is greater than $\pi/2$. It is a simple matter to compute the angle θ at which φ reaches its extrema. We get the minimum of φ at $\theta = \arccos(-e)$ and the maximum at $\theta = 2\pi - \arccos(-e)$. Thus the smallest φ equals $\varphi_{\min} = \pi - \arccos(-e) = \arccos(e)$ and the highest φ equals $\varphi_{\max} = \arccos(-e)$. From the analysis of (8), for the interval $0 \leq \theta < 2\pi$ of figure 3, the ambiguity of the arctan-function in equation (8) is resolved as follows.

$$\varphi = \begin{cases} \frac{\pi}{2} + \arctan\left(\frac{\sin \theta}{e + \cos \theta}\right) - \theta & \text{for } 0 \leq \theta < \arccos(-e) \\ \frac{3\pi}{2} + \arctan\left(\frac{\sin \theta}{e + \cos \theta}\right) - \theta & \text{for } \arccos(-e) \leq \theta < 2\pi - \arccos(-e) \\ \frac{5\pi}{2} + \arctan\left(\frac{\sin \theta}{e + \cos \theta}\right) - \theta & \text{for } 2\pi - \arccos(-e) \leq \theta < 2\pi. \end{cases}$$

Figure 3: Angle φ between $\mathbf{r}$ and $\mathbf{v}$ as function of θ (for $e = 0.6$)



7 Conclusion

Starting with Kepler's first and second laws it is essentially a three line derivation to obtain Newton's gravitational law. In addition, from the computation one essentially gets Hamilton's hodograph as side effect. The computation of the speed as function of the angle ϑ is a straight-forward task as well as the dependency of ϑ from the true anomaly θ (and its inversion). Using elliptic functions one can also express elegantly the speed and invert it, i.e. express ϑ as function of the speed. The new expression of the speed (eqn.(6)) is essentially an application of the Gauss transform ³ to the elliptic function $\text{dn}(u, k) + k \cdot \text{cn}(u, k)$. The Gauss transform changes the modulus k of the elliptic function to $2\sqrt{k}/(1 + k)$.

³The Gauss transform is the inverse of the Landen transform.

A Elliptic Functions used and Derivation of an Identity

We use the standard notation of Gudermann (see [3], p.494):

$$\begin{aligned}\sin \operatorname{am}(u, k) &= \operatorname{sn}(u, k) \\ \cos \operatorname{am}(u, k) &= \operatorname{cn}(u, k) \\ \sqrt{1 - k^2 \operatorname{sn}^2(u, k)} &= \operatorname{dn}(u, k) .\end{aligned}$$

We now derive a formula which leads to equation (6). Using the transformation formulas 8.152 from [1], p.915 (the 7-th row) – where $k' = \sqrt{1 - k^2}$, $k_1 = \frac{1-k'}{1+k'}$, and $u_1 = (1 + k')u$ – we find

$$\operatorname{dn}(u_1, k_1) + k_1 \operatorname{cn}(u_1, k_1) = \frac{1 - (1 - k')\operatorname{sn}^2(u, k)}{\operatorname{dn}(u, k)} + \frac{1 - k'}{1 + k'} \frac{1 - (1 + k')\operatorname{sn}^2(u, k)}{\operatorname{dn}(u, k)}$$

which leads to

$$\operatorname{dn}(u_1, k_1) + k_1 \operatorname{cn}(u_1, k_1) = \frac{2}{1 + k'} \operatorname{dn}(u, k)$$

If we set $k_1 = e$ it follows that $k' = \frac{1-e}{1+e}$, $k = \frac{2\sqrt{e}}{1+e}$, and $\frac{2}{1+k'} = 1 + e$ hence

$$\operatorname{dn}(u_1, e) + e \operatorname{cn}(u_1, e) = (1 + e) \operatorname{dn}\left(\frac{1 + e}{2} u_1, \frac{2\sqrt{e}}{1 + e}\right)$$

To avoid collision with the different use of u above we replace u_1 by z and get

$$\frac{\operatorname{dn}(z, e) + e \operatorname{cn}(z, e)}{\sqrt{1 - e^2}} = \sqrt{\frac{1 + e}{1 - e}} \operatorname{dn}\left(\frac{1 + e}{2} z, \frac{2\sqrt{e}}{1 + e}\right) .$$

References

- [1] I.S.Gradsteyn, I.M.Ryzhik, "Tables of Integrals, Series, and Products", Academic Press, Sixth Printing, 1979.
- [2] W.R.Hamilton, "New mode of geometrically conceiving, and of expressing in symbolical language, the Newtonian law of attraction, and the mathematical problem of determining the orbits and perturbations of bodies which are governed in their motions by that law", Proceedings of the Royal Irish Academy, Vol. III, 1847, pp. 344-353.
- [3] E.T.Whittaker, G.N.Watson, "A course of modern analysis", fourth edition, Cambridge, reprinted 1969.